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LOCALLY SUPERSYMMETRIC σ-MODEL WITH WESS-ZUMINO TERM IN TWO DIMENSIONS AND CRITICAL DIMENSIONS FOR STRINGS

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We construct an $N=1$ locally supersymmetric σ-model with a Wess-Zumino term coupled to supergravity in two dimensions. If one takes the σ-model manifold to be the product of d-dimensional Minkowski space M^d and a group manifold G, and if the radius of G is quantized in appropriate units of the string tension, then the model describes a Neveu-Schwarz-Ramond (NSR)-type string moving on $M_d \times G$ (Our model generalizes earlier work of refs. [1,2] which do not contain a Wess-Zumino term and that of refs [5,6] which is not locally supersymmetric) The zweibein and the gravitino field equations yield constraints which generalize those of the NSR model to the case of a non-abelian group manifold In particular, the fermionic constraint contains a new term trilinear in the fermionic fields We quantize the theory in the light-cone gauge and derive the critical dimensions. We compute the mass spectrum of a closed string moving on $M_d \times G$ and show that massless fermions do not arise for non-abelian G for the spinning string, in agreement with the result of Friedan and Shenker [22]

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1. Introduction

Some time ago, Deser and Zumino [1] and Brink, di Vecchia and Howe [2] constructed the coupling of ten scalar supersymmetric multiplets to $d=2$ supergravity. They showed that the theory is conformally invariant and describes the $d=10$ Neveu-Schwarz-Ramond [3] string model [15]. From the $d=2$ point of view this theory is a locally supersymmetric σ-model in which the scalar manifold is a $d=10$ flat Minkowski space-time, $M_{10} = ISO(9,1)/SO(9,1)$. In this paper we consider the generalization of this model in which the scalars parametrize an arbitrary riemannian manifold. In such a generalization, a Wess-Zumino term [4–6] is coupled to $d=2$ supergravity. This model furnishes a covariant description of a string moving in curved space. In particular when this space is $M_d \times G$ where M_d is d-dimensional Minkowski space-time and G is a compact group manifold, the characteristic size of G is quantized in units of the string tension [7, 8b], while restrictions on d arise, due to the requirement of Lorentz invariance in M_d, of the quantized theory.

These results are relevant for the reduction of the critical dimensions in which the string theory can be consistently quantized. Recently, in refs. [7] and [8b] it was shown that the critical dimension for the bosonic string is given by

$$d = 26 - \frac{d_G}{1 + c_A/2|k|} \qquad \text{(bosonic string)}, \tag{1.1}$$

where d_G is the dimension of the group G, k is an integer and c_A is the eigenvalue of the second Casimir operator of G in the adjoint representation (see eq. (1.3)). Extending this result, in ref. [7] it was conjectured that the critical dimension for the fermionic string is given by

$$d = 10 - \frac{2}{3}\frac{d_G}{1 + c_A/2|k|} - \tfrac{1}{3}d_G \qquad \text{(spinning string)}. \tag{1.2}$$

The value of c_A for Lie groups is given by

$$\begin{array}{cccccccccc} G & SU(n) & SO(2n+1) & Sp(n) & SO(2n) & G_2 & F_4 & E_6 & E_7 & E_8 \\ c_A & 2n & 4n-2 & 2n+2 & 4n-4 & 8 & 18 & 24 & 36 & 60 \end{array}. \tag{1.3}$$

In this paper, starting from our locally supersymmetric action (see eq. (2.1)) we quantize the theory in the light-cone gauge, rederive (1.1) and verify (1.2). We emphasize that the main body of this paper deals with (i) the construction of the $N=1$, $d=2$ locally supersymmetric σ-model with Wess-Zumino term and (ii) the derivation of the critical dimension formula for the fermionic superstring based on this model.

The case of $k=1$ is special, since, as was shown by Witten [9], in this case the scalars of the non-linear σ-model become equivalent to free fermions. For $k=1$, the solution to (1.1) includes $G = SU(n)$, $SO(2n)$, E_6, E_7, E_8 or any product of these groups with*

$$r = 26 - d, \tag{1.4}$$

where r is the rank of G**. As is well known, for $d = 10$, $E_8 \times E_8$ is a solution of (1.4) which was used in the construction of the heterotic string [12].

For $k = 1$, assuming that G is simple, the unique solution to (1.2) is***

$$\begin{aligned} d &= 8, \quad G = SO(3), \\ d &= 6, \quad G = SU(3), \\ d &= 5, \quad G = SO(5), \\ d &= 3, \quad G = SU(4). \end{aligned} \quad \text{(spinning string)} \tag{1.5}$$

(Note that $d = 4$ is not included for any *simple* G.) All these solutions refer to classically and quantum mechanically consistent *free* string theories. At present it is not clear how to formulate consistent interacting string theories based on these solutions.

This paper is organized as follows. In sect. 2 we construct the $N = 1$ locally supersymmetric action with a Wess-Zumino term. In sect. 3 we derive the field equations and constraints following from the action. In sect. 4 we quantize the system and derive the critical dimension formula by the requirement of Lorentz invariance in M_d. In sect. 5 we discuss the spectrum of a closed string moving on $M_d \times G$. For the bosonic case it coincides with that of the string which compactifies on r-tori where r is the rank of a simply laced group. Various aspects of this phenomenon have been discussed by several authors [7, 8a, 19, 20]. In the fermionic case, we find that massless fermions do not arise for nonabelian groups, in agreement with the result of Friedan and Shenker [22]. Finally in sect. 6 we discuss some of the open problems.

* For $k > 1$, some of the solutions to (1.1) are: $d = 7$, E_7 ($k = 3$); $d = 8$, SU(5) ($k = 15$); $d = 6$, SU(7) ($k = 5$).

** For a complete discussion see Goddard, Nahm and Olive (and other papers) in ref. [8a]. (It has recently been shown [10] that for arbitrary k the $O(N)$ σ-model is equivalent to free particles obeying parastatistics of order k. Regarding the question of parastatistics in strings see also ref [11] where it is shown that $d = 2 + 8/q$ for a free fermionic theory exhibiting parastatistics of order q For $q = 2$ this gives $d = 6$.)

*** For $k > 1$, the unique solution (for simple G) is: $d = 8$, SU(2) ($k = 2$), $d = 4$, SO(5) ($k = 2$), SU(3) ($k = 5$); $d = 2$, SO(5) ($k = 7$)

2. A locally supersymmetric action with the Wess-Zumino term

We construct the action of an $N=1$ locally supersymmetric σ-model coupled to supergravity in two dimensions, where the scalars of the σ-model parametrize an arbitrary riemannian manifold M. $N=1$ supergravity in two dimensions contains a zweibein e_μ^a and a gravitino ψ_μ. As is well known, in two dimensions these fields do not describe physical degrees of freedom. Nevertheless they play an important role in that their field equations yield constraints on the scalars and spinors of the σ-model.

The full action reads as follows:

$$
\begin{aligned}
e^{-1}\mathcal{L} = \frac{1}{2\pi\alpha'}\Big[& -\tfrac{1}{2} g^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j g_{ij} - \tfrac{1}{2} i\bar{\chi}^i\gamma^\mu\left(\partial_\mu\chi^j + \Gamma^j_{kl}\,\partial_\mu\phi^k\chi^l\right)g_{ij} \\
& + \bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\,\partial_\nu\phi^j g_{ij} - \tfrac{1}{4}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\bar{\chi}^i\chi^j g_{ij} - \tfrac{1}{12}R_{ijkl}\bar{\chi}^i\chi^k\bar{\chi}^j\chi^l \\
& - e^{-1}\frac{k}{8\pi}\varepsilon^{\mu\nu}b_{ij}\,\partial_\mu\phi^i\,\partial_\nu\phi^j - \frac{ik}{16\pi}T_{ijk}\bar{\chi}^i\gamma^\mu\gamma_5\chi^j\,\partial_\mu\phi^k \\
& + \frac{k}{64\pi}T_{ilj;k}\bar{\chi}^i\chi^k\bar{\chi}^j\gamma_5\chi^l - \frac{k^2}{512\pi^2}g^{mn}T_{ikm}T_{jln}\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_5\chi^l \\
& + \frac{ik}{48\pi}T_{ijk}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\bar{\chi}^j\gamma_\nu\gamma_5\chi^k\Big],
\end{aligned}
$$

$$\mu = 0,1, \qquad i = 1,\dots,\dim \mathrm{M} \qquad (2.1)$$

and is invariant under the following transformations:

$$
\begin{aligned}
\delta e_\mu^a &= 2i\bar{\varepsilon}\gamma^a\psi_\mu - \Lambda e_\mu^a, \\
\delta\psi_\mu &= \left(\partial_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\varepsilon + i\gamma_\mu\eta - \tfrac{1}{2}\Lambda\psi_\mu, \\
\delta\phi^i &= -\bar{\varepsilon}\chi^i, \\
\delta\chi^i &= -i\gamma^\mu\left(\partial_\mu\phi^i + \bar{\psi}_\mu\chi^i\right)\varepsilon - \Gamma^i_{jk}\delta\phi^j\chi^k + \frac{k}{16\pi}g^{il}T_{ljk}\left(\bar{\chi}^j\gamma_5\chi^k\right)\varepsilon + \tfrac{1}{2}\Lambda\chi^i, \qquad (2.2)
\end{aligned}
$$

where $\varepsilon(\sigma,\tau)$, $\eta(\sigma,\tau)$ and $\Lambda(\sigma,\tau)$ are the supersymmetry, conformal supersymmetry and Weyl scale transformation parameters, respectively. In (2.1) and (2.2) we have used the following definitions and conventions. The scalars ϕ^i ($i=1,\dots\dim\mathrm{M}$) parametrize a riemannian manifold, M, with metric $g_{ij}(\phi)$. The spinors χ^i and ψ_μ

are two-component Majorana*. The parameter α' is the coupling constant of the σ-model and the coefficient k in front of the Wess-Zumino term is defined such that the path integral corresponding to (2.1) is well defined [9] (and last reference in [6]). Thus k is an integer. Our Riemann tensor is defined as $R^i{}_{jkl} = \partial_l \Gamma^i_{jk} + \cdots$. Note that $\gamma^\mu = \gamma^a e^\mu_a (\mu, a = 0,1)$. The 3-form T_{ijk} is closed and is the curl of a second-rank antisymmetric tensor $b_{ij}(\phi) = -b_{ji}(\phi)$, which is a function of the scalars ϕ:

$$T_{ijk} = \partial_i b_{jk} + \partial_k b_{ij} + \partial_j b_{ki} . \tag{2.3}$$

The spin connection, ω_μ^{ab}, contains the contorsion tensor $K_{\mu ab} = 2i\bar{\psi}_a \psi^\mu \psi_b$. The term in $\delta\chi^i$ containing the Christoffel symbol Γ^i_{jk} has been added so that when it is taken to the left-hand side, $(\delta\chi^i + \Gamma^i_{jk}\delta\phi^j\chi^k)$ transforms as a vector on M, as it should since the remaining terms on the right-hand side have the same transformation property. Finally, the notation ; is standard for riemannian covariant derivatives.

In order to derive (2.1) and (2.2) we proceed in two steps. First we generalize the action and transformation rules given in refs. [1] and [2] to the case of an arbitrary riemannian manifold, M, with metric $g_{ij}(\phi)$, as follows:

$$e^{-1}\mathcal{L}_1 = \frac{1}{2\pi\alpha'}\Big[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij} - \tfrac{1}{2}i\bar{\chi}^i\gamma^\mu\big(\partial_\mu\chi^j + \Gamma^k_{kl}\partial_\mu\phi^k\chi^l\big)g_{ij}$$
$$+\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\partial_\nu\phi^j g_{ij} - \tfrac{1}{4}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\bar{\chi}^I\chi^J g_{ij} - \tfrac{1}{12}R_{ijkl}\bar{\chi}^i\chi^k\bar{\chi}^j\chi^l\Big] \tag{2.4}$$

and

$$\delta e_\mu^a = 2i\bar{\varepsilon}\gamma^a\psi_\mu - \Lambda e_\mu^a ,$$
$$\delta\psi_\mu = \big(\partial_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\big)\varepsilon + i\gamma_\mu\eta - \tfrac{1}{2}\Lambda\psi_\mu ,$$
$$\delta\phi^i = -\bar{\varepsilon}\chi^i ,$$
$$\delta\chi^i = -i\gamma^\mu\big(\partial_\mu\phi^i + \bar{\psi}_\mu\chi^i\big)\varepsilon - \Gamma^i_{jk}\delta\phi^j\chi^k + \tfrac{1}{2}\Lambda\chi^i . \tag{2.5}$$

Remarkably we find that without any further modifications the covariantized action (2.4) is already fully invariant under the local supersymmetry and scale transformations given in (2.5). In fact a large class of variations are those which arise in either the model of refs. [1] and [2] $(g_{\mu\nu} \neq \eta_{\mu\nu}, g_{ij} = \eta_{ij})$ provided that in the

* We use the following conventions: Our metric is $\eta_{ab} = \mathrm{diag}(-1, +1)$. The $d = 2$ gamma matrices γ^a $(a = 0,1)$ are $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ and $\gamma_5 = -\sigma_3$. We take $\varepsilon^{01} = -1$, $\varepsilon_{01} = +1$ and $\bar{\chi} = \chi^T\gamma^0$. In two dimensions the following Fierz relations hold: $\psi\bar{\lambda} = -\frac{1}{2}(\bar{\lambda}\psi + (\bar{\lambda}\gamma_5\psi)\gamma_5 - (\bar{\lambda}\gamma_\mu\psi)\gamma_\mu)$ and $\psi(\bar{\lambda}\chi) = -\lambda(\bar{\chi}\psi) - \chi(\bar{\psi}\lambda)$. Some further useful relations are: $\gamma_\mu\gamma_\nu = -g_{\mu\nu} + e^{-1}\varepsilon_{\mu\nu}\gamma_5$, $e = \det e_\mu^a$ and $\varepsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$.

variation of the action the derivatives are covariantized with respect to M (e.g. $\partial_\mu \chi^i \to \partial_\mu \chi^i + \Gamma^i_{jk} \partial_\mu \phi^j \chi^k$) or the globally supersymmetric σ-model [13] ($g_{\mu\nu} = \eta_{\mu\nu}$, $g_{ij} \neq \eta_{ij}$). Therefore one only has to check the cancellation of the new variations which do not fall into these two classes. Most cancellations are trivial, the only non-trivial ones being those arising from the variation of the zweibein and χ^i in the $R\chi^4$ term. These variations give a vanishing result:

$$-\tfrac{1}{6} i \bar{\varepsilon} \gamma^\lambda \psi_\lambda R_{ijkl} \bar{\chi}^i \chi^k \bar{\chi}^j \chi^l + \tfrac{1}{6} R_{ijkl} \bar{\chi}^i \chi^k \bar{\chi}^j \left(i \gamma^\nu \varepsilon \bar{\psi}_\nu \chi^l \right)$$
$$+ \tfrac{1}{6} R_{ijkl} \bar{\chi}^i \chi^k \bar{\chi}^l \left(i \gamma^\nu \varepsilon \bar{\psi}_\nu \chi^j \right) = 0 . \tag{2.6}$$

To prove this, one must Fierz rearrange χ^j and χ^l in the second and third term and use the fact that

$$R_{ijkl} \bar{\chi}^i \chi^k \bar{\chi}^j \gamma_5 \chi^l = 0 = R_{ijkl} \bar{\chi}^i \chi^k \bar{\chi}^j \gamma_\mu \chi^l . \tag{2.7}$$

This identity is easily proven by noting that R_{ijkl} is symmetric, whereas the χ terms are antisymmetric in the pair interchange $(ij) \leftrightarrow (kl)$. We thus conclude that the action given in (2.4) is invariant under (2.5).

We now consider the extension of (2.4), (2.5) by adding a Wess-Zumino term in a locally supersymmetric manner. A globally supersymmetric Wess-Zumino term has been constructed in refs. [5,6]. In the case of local supersymmetry we leave the transformation rules (2.5) intact except for $\delta\chi^i$ which we modify to read

$$\delta \chi^i = -i \gamma^\mu \left(\partial_\mu \phi^i + \bar{\psi}_\mu \chi^i \right) \varepsilon - \Gamma^i_{jk} \delta \phi^j \chi^k + \frac{k}{16\pi} g^{il} T_{ljk} \left(\bar{\chi}^j \gamma_5 \chi^k \right) \varepsilon . \tag{2.8}$$

Note that the k-dependent term in $\delta\chi^i$ is precisely the one which occurs in the globally supersymmetric model. To obtain an action which is invariant under the local supersymmetries (2.5), (2.8), we first covariantize the globally supersymmetric Wess-Zumino action [5,6] with respect to the two-dimensional space-time. This yields

$$e^{-1} \mathcal{L}_2 = -e^{-1} \frac{k}{8\pi} \varepsilon^{\mu\nu} b_{ij} \, \partial_\mu \phi^i \, \partial_\nu \phi^j - \frac{ik}{16\pi} T_{ijk} \bar{\chi}^i \gamma^a \gamma^5 \chi^j \, \partial_\mu \phi^k \, e^\mu_a$$
$$+ \frac{k}{64\pi} T_{ilj;k} \bar{\chi}^i \chi^k \bar{\chi}^j \gamma_5 \chi^l - \frac{k^2}{512\pi^2} g^{mn} T_{ikm} T_{jln} \bar{\chi}^i \gamma_5 \chi^k \bar{\chi}^j \gamma_5 \chi^l . \tag{2.9}$$

We now consider the new variations in $\mathcal{L}_1 + \mathcal{L}_2$, given in (2.4) and (2.9), which do not arise in the globally supersymmetric Wess-Zumino term of refs. [5] and [6]. One class of variations are the terms proportional to T^2 coming from the variation of the determinant e and of χ^i in the $T^2\chi^4$ term. These variations are

$$-\frac{1}{2\pi\alpha'}\left[\frac{k^2}{512\pi^2}T_{ikm}T_{jl}{}^m\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_5\chi^l\left(2i\bar{\varepsilon}\gamma^\lambda\psi_\lambda\right)\right.$$
$$\left.+\frac{k^2}{128\pi^2}T_{ikm}T_{jl}{}^m\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_5\left(-i\gamma^\nu\varepsilon\bar{\psi}_\nu\chi^l\right)\right]. \tag{2.10}$$

The Fierz rearrangement of $\chi^l\bar{\chi}^j$ in the second term yields two terms one of which cancels the first term. The final result is

$$\frac{-ik^2}{512\alpha'\pi^3}T_{ikm}T_{jl}{}^m\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_\lambda\chi^l\bar{\varepsilon}\gamma^\nu\gamma_5\gamma^\lambda\psi_\nu. \tag{2.11}$$

In order to cancel this we add to the action the following new term:

$$e^{-1}\mathcal{L}_3=\frac{ik}{96\pi^2\alpha'}T_{ijk}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\bar{\chi}^j\gamma_\nu\gamma_5\chi^k. \tag{2.12}$$

One can easily show by appropriate Fierz rearrangements that all the χ's in (2.12) contribute with the same weight to $\delta\mathcal{L}_3$. In particular, the T-dependent variation of the χ's in (2.12) gives

$$3\times\frac{ik}{96\pi^2\alpha'}\left(T_{ijk}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\right)\left(\frac{k}{16\pi}T_{imn}\bar{\chi}^m\gamma_5\chi^n\bar{\chi}^j\gamma_\nu\gamma_5\chi^k\varepsilon\right), \tag{2.13}$$

which exactly cancels (2.11).

Remarkably we now find that with the addition of $\mathcal{L}_3$ given in (2.12) the full lagrangian $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$ given in (2.4), (2.9), (2.12) or (2.1) is invariant under the local supersymmetries (2.5), (2.8) or (2.2) without any further modifications. We have checked this by a straightforward but tedious calculation which will not be reproduced.

3. Field equations and constraints

We now study the field equations which follow from the action, (2.1). As mentioned in the introduction the scalar manifold is taken to be $\mathrm{M}_d\times\mathrm{G}$. We shall see that for a particular relation among the parameters of the theory (see eq. (3.7), the field equations for ϕ^i and χ^i will be completely integrable. The field equations for e_μ^a and ψ_μ lead to constraints on ϕ^i and χ^i which generalize those of the Neveu-Schwarz-Ramond model. These constraints will play an important role in determining the critical dimension for strings in curved space, as we shall show in the next section.

We first choose co-ordinates appropriate to the product structure $M_d \times G$:

$$\phi^i = \begin{pmatrix} x^\alpha \\ y^I \end{pmatrix}, \qquad \chi^i = \begin{pmatrix} \lambda^\alpha \\ \chi^I \end{pmatrix}, \qquad \alpha = 0,1,\dots,d-1, \qquad I = 1,\dots,d_G. \quad (3.1a)$$

Correspondingly

$$g_{ij} = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & g_{IJ}(y) \end{pmatrix}, \qquad b_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2\pi\alpha'}{R^2} b_{IJ}(y) \end{pmatrix}, \quad (3.1b)$$

where $\eta_{\alpha\beta}$ is the usual metric in M_d and g_{IJ} is the $G \times G$ invariant metric on G. Note that we have introduced an additional parameter R, which is associated with the characteristic size of the compact group manifold G. In terms of the coordinates x^α and y^I the lagrangian (2.1) splits into two parts which are separately invariant. One part, $\mathcal{L}(M_d)$, depends only on x^α and has exactly the same form as the lagrangian given in refs. [1] and [2]

$$e^{-1}\mathcal{L}(M_d) = \frac{1}{2\pi\alpha'}\Big[-\tfrac{1}{2} g^{\mu\nu}\,\partial_\mu x^\alpha\,\partial_\nu x^\beta\,\eta_{\alpha\beta} - \tfrac{1}{2} i\bar{\lambda}^\alpha \gamma^\mu\,\partial_\mu \lambda^\beta\,\eta_{\alpha\beta}$$
$$+ \bar{\psi}_\mu \gamma^\nu \gamma^\mu \lambda^\alpha \big(\partial_\nu x^\beta\big)\eta_{\alpha\beta} - \tfrac{1}{4}\bar{\psi}_\mu \gamma^\nu \gamma^\mu \psi_\nu \bar{\lambda}^\alpha \lambda^\beta \eta_{\alpha\beta}\Big]. \quad (3.2)$$

The other part $\mathcal{L}(G)$ depends only on the group co-ordinates y^I. On a group manifold G we can take T_{IJK} to be

$$T_{IJK} = \frac{1}{R} f_{abc} L_I^a L_J^b L_K^c = -\frac{1}{R} f_{abc} R_I^a R_J^b R_K^c \equiv \frac{1}{R} f_{IJK}, \quad (3.3)$$

where $f_{ab}{}^c$ are the structure constants of the group G and $L_I^a(\phi)$ are the left-invariant basis elements on the group manifold G, which are defined by

$$g^{-1}\,\partial_\mu g = \frac{1}{R} L_I^a T_a\,\partial_\mu y^I, \qquad g\,\partial_\mu g^{-1} = -\frac{1}{R} R_I^a T_a\,\partial_\mu y^I. \quad (3.4)$$

Here g is a group element and T^a are the antihermitian generators of the Lie algebra of G. From (3.3) it follows that T_{IJK} is an invariant 3-form on G and hence that b_{IJ} is an invariant 2-form (up to a total derivative).

Using (3.3) and the following relations [14, 13]

$$R_{IJKL} = \frac{-1}{4R^2} f_{IJ}{}^M f_{KLM}, \quad (3.5a)$$

$$R_{IJKL}\bar{\chi}^I \chi^K \bar{\chi}^J \chi^L = 3R_{IJKL}\bar{\chi}^I \gamma_5 \chi^K \bar{\chi}^J \gamma_5 \chi^L, \quad (3.5b)$$

the Lagrangian $\mathcal{L}(\mathrm{G})$ now reads

$$e^{-1}\mathcal{L}(\mathrm{G})=\frac{-1}{4\pi\alpha'}\,\partial_\mu y^I\,\partial_\nu y^J\left(g^{\mu\nu}g_{IJ}+\frac{k\alpha'}{2R^2}e^{-1}\varepsilon^{\mu\nu}b_{IJ}\right)$$
$$-\frac{i}{4\pi\alpha'}\bar{\chi}^I\gamma^\mu\left[\partial_\mu\chi^J+\left(\Gamma^J_{KL}-\frac{k\alpha'}{4R^3}\gamma_5 f^J_{KL}\right)\partial_\mu y^K\chi^L\right]g_{IJ}$$
$$+\frac{1}{32\pi\alpha' R^2}f_{IJ}{}^M f_{KLM}\left(1-\frac{\alpha'^2k^2}{4R^4}\right)\bar{\chi}^I\gamma_5\chi^K\bar{\chi}^J\gamma_5\chi^L$$
$$+\frac{1}{2\pi\alpha'}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^I\,\partial_\nu y^J g_{IJ}-\frac{1}{8\pi\alpha'}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\bar{\chi}^I\chi^J g_{IJ}$$
$$+\frac{ik}{48\pi R^3}f_{IJK}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^I\bar{\chi}^J\gamma_\nu\gamma_5\chi^K\,. \tag{3.6}$$

We see that for

$$R^2=\tfrac{1}{2}|k|\alpha' \tag{3.7}$$

the quartic χ^4 terms in (3.6) cancel. Without loss of generality we shall always take k to be positive. Using (3.7), in the superconformal gauge

$$e^a_\mu=f(\sigma,\tau)\,\delta^a_\mu\,,\qquad \psi_\mu=\gamma_\mu\lambda(\sigma,\tau) \tag{3.7$'$}$$

the field equations following from (3.2) and (3.6) are

$$A_1=\eta^{\mu\nu}\,\partial_\mu\partial_\nu x^\alpha=0\,,\qquad A_2=\gamma^\mu\,\partial_\mu\lambda^\alpha=0\,, \tag{3.8a, b}$$

$$A_3=\eta^{\mu\nu}\,\partial_\mu\partial_\nu y^I+\partial_\mu y^J\,\partial_\nu y^k\left(g^{\mu\nu}\Gamma^I_{JK}-\frac{1}{2R}\varepsilon^{\mu\nu}f_{JK}{}^I\right)=0\,, \tag{3.8c}$$

$$A_4=\gamma^\mu\,\partial_\mu\chi^I+\gamma^\mu\left(\Gamma^I_{JK}-\frac{1}{2R}\,f_{JK}{}^I\gamma_5\right)\partial_\mu y^J\chi^K=0\,, \tag{3.8d}$$

$$A_5=\gamma^\nu\gamma^\mu\lambda_\alpha\,\partial_\nu x^\alpha+\gamma^\nu\gamma^\mu\chi^I\,\partial_\nu y^J g_{IJ}+\frac{i}{12R}f_{IJK}\gamma^\nu\gamma^\mu\chi^I\bar{\chi}^J\gamma_\nu\gamma_5\gamma^K=0\,, \tag{3.8e}$$

$$A_6=\eta_{\alpha\beta}\left(-\partial_\mu x^\alpha\,\partial_\nu x^\beta+\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\,\partial_\rho x^\alpha\,\partial_\sigma x^\beta\right)-\tfrac{1}{2}i\bar{\lambda}^\alpha\gamma_\nu\,\partial_\mu\lambda_\alpha$$
$$+g_{IJ}\left(-\partial_\mu y^I\,\partial_\nu y^J+\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\,\partial_\rho y^I\,\partial_\sigma y^J\right)-\tfrac{1}{2}i\bar{\chi}^I\gamma_\nu\,\partial_\mu\chi^J g_{IJ}$$
$$-\tfrac{1}{2}i\bar{\chi}^I\gamma_\nu\left(\Gamma^J_{MN}-\frac{1}{2R}\gamma_5 f_{MN}{}^J\right)\partial_\mu y^M\chi^N g_{IJ}=0\,. \tag{3.8f}$$

Note that $A_1 = A_2 = A_3 = A_4 = 0$ are the physical field equations, while $A_5 = A_6 = 0$ are constraint equations. The field equation $A_3 = 0$ corresponds to the variation $\delta I/\delta y^I - \Gamma^K_{IJ}(\delta I/\delta\chi^K)\chi^J = 0$ to ensure covariance. This variation gives the result (3.8c) plus a term which vanishes upon use of the χ^I field equation. Note that the resulting equation, $A_3 = 0$, does not depend on χ^I. Also in (3.8f) the χ^I field equation has been used.

It is convenient to rewrite (3.8c) in terms of the bosonic part of the currents associated with the right and left translations on the group manifold. These currents are defined by

$$J^{\mu a}_{\mathrm{R}} = \tfrac{1}{2}(\eta^{\mu\nu} + \varepsilon^{\mu\nu})\,\partial_\nu y^I L^a_I, \tag{3.9a}$$

$$J^{\mu a}_{\mathrm{L}} = \tfrac{1}{2}(\eta^{\mu\nu} - \varepsilon^{\mu\nu})\,\partial_\nu y^I R^a_I. \tag{3.9b}$$

Furthermore, the following relations will be useful in simplifying the field equations [14]

$$g_{IJ} = L^a_I L^a_J = R^a_I R^a_J,$$

$$\Gamma^I_{JK} = -L^a_K\,\partial_J L^I_a + \frac{1}{2R} f_{JK}{}^I = -R^a_K\,\partial_J R^I_a - \frac{1}{2R} f_{JK}{}^I,$$

$$\partial_I L^a_J - \partial_J L^a_I + \frac{1}{R} f_{IJ}{}^K L^a_K = 0 = \partial_I R^a_J - \partial_J R^a_I - \frac{1}{R} f_{IJ}{}^K R^a_K. \tag{3.10}$$

Using (3.9) and (3.10), we can rewrite (3.8c) as

$$\partial_\mu J^{\mu a}_{\mathrm{R}} = 0, \qquad \partial_\mu J^{\mu a}_{\mathrm{L}} = 0. \tag{3.11a,b}$$

To simplify $A_4 = A_5 = A_6 = 0$ we now define the components of χ^I in the left and right invariant basis elements as follows:

$$\chi^a = \chi^I L^a_I, \qquad \tilde\chi^a = \chi^I R^a_I. \tag{3.12}$$

It is not difficult to show that the supersymmetry variation of $J^{\mu a}_{\mathrm{R}}$ contains $\chi^a_{\mathrm{L}} = \tfrac{1}{2}(1+\gamma_5)\chi^a$ and that of $J^{\mu a}_{\mathrm{L}}$ contains $\tilde\chi^a_{\mathrm{R}} = \tfrac{1}{2}(1-\gamma_5)\tilde\chi^a$. Thus it is natural to express equations $A_4 = A_5 = A_6 = 0$ in terms of χ^a_{L} and $\tilde\chi^a_{\mathrm{R}}$. Again using (3.9) and (3.10), from (3.8d, e, f) it now follows that

$$\gamma^\mu\,\partial_\mu\chi^a_{\mathrm{L}} = 0, \qquad \gamma^\mu\,\partial_\mu\tilde\chi^a_{\mathrm{R}} = 0, \tag{3.13a,b}$$

$$(\eta^{\mu\nu} + \varepsilon^{\mu\nu})\left(\lambda_{\mathrm{L}\alpha}\,\partial_\nu x^\alpha + \chi^a_{\mathrm{L}} J^a_{\nu\mathrm{R}} - \frac{i}{12R} f_{abc}\chi^a_{\mathrm{L}}\bar\chi^b_{\mathrm{L}}\gamma_\nu\chi^c_{\mathrm{L}}\right) = 0, \tag{3.14a}$$

$$(\eta^{\mu\nu} - \varepsilon^{\mu\nu})\left(\lambda_{\mathrm{R}\alpha}\,\partial_\nu x^\alpha + \tilde\chi^a_{\mathrm{R}} J^a_{\nu\mathrm{L}} - \frac{i}{12R} f_{abc}\tilde\chi^a_{\mathrm{R}}\bar{\tilde\chi}^b_{\mathrm{R}}\gamma_\nu\tilde\chi^c_{\mathrm{R}}\right) = 0, \tag{3.14b}$$

$$\begin{aligned}\eta_{\alpha\beta}&\left(-\partial_\mu x^\alpha\,\partial_\nu x^\beta + \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\,\partial_\rho x^\alpha\,\partial_\sigma x^\beta\right) - \tfrac{1}{2}i\bar\lambda^\alpha_{\mathrm{L}}\gamma_\nu\,\partial_\mu\lambda_{\mathrm{L}\alpha}\\ &- \tfrac{1}{2}i\bar\lambda^\alpha_{\mathrm{R}}\gamma_\nu\,\partial_\mu\lambda_{\mathrm{R}\alpha} + \eta_{\alpha\beta}\left(-\partial_\mu y^I\,\partial_\nu y^J + \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho y^I\,\partial_\sigma y^J\right)\\ &- \tfrac{1}{2}i\bar\chi^a_{\mathrm{L}}\gamma_\nu\,\partial_\mu\chi^a_{\mathrm{L}} - \tfrac{1}{2}i\bar{\tilde\chi}^a_{\mathrm{R}}\gamma_\nu\,\partial_\mu\tilde\chi^a_{\mathrm{R}} = 0.\end{aligned} \tag{3.15}$$

Note that the last terms in (3.8d, f) have dropped out, due to the fact that we have expressed $\chi^I_{L,R}$ in the appropriate basis.

In summary, the field equations following from the action (3.2), (3.6) are given by (3.8a, b), (3.11a, b) and (3.13a, b), while the constraints are given by (3.14) and (3.15). In particular, note the presence of the χ^3 terms in (3.14) which are necessary for the closure of the super-Virasoro algebra. We will use these equations as a starting point for the quantization of the system in the next section.

4. The critical dimension formula for strings moving on $M^d \times G$

We now solve the free field equations given in (3.8), (3.11) and (3.13). Next we quantize the system subject to the constraints given in (3.14) and (3.15). The requirement of Lorentz invariance of the quantized theory in M_d puts restrictions on the dimensions d and d_G of M_d and the group manifold G, respectively. These restrictions have recently been obtained for the bosonic string in refs. [7] and [8b] and conjectured for the fermionic string in ref. [7]. In this section we shall restrict ourselves to closed strings. The case of open string can be treated similarly.

The solutions to the field equations are given by (setting $2\alpha' = 1$)*

$$x^\alpha(\tau,\sigma) = q^\alpha + p^\alpha\tau + \tfrac{1}{2}i \sum_{n\neq 0} \frac{1}{n}\left(\alpha^\alpha_n e^{-2in(\tau+\sigma)} + \tilde{\alpha}^\alpha_n e^{-2in(\tau-\sigma)}\right), \tag{4.1a}$$

$$J^a_{+R} = \sqrt{2} \sum_{n=-\infty}^{\infty} \beta^a_n e^{-2in(\tau+\sigma)}, \tag{4.1b}$$

$$J^a_{-L} = \sqrt{2} \sum_{n=-\infty}^{\infty} \tilde{\beta}^a_n e^{-2in(\tau-\sigma)}, \tag{4.1c}$$

$$\lambda^{\alpha(1)} = \sum_{n=-\infty}^{\infty} \tilde{d}^\alpha_n e^{-2in(\tau-\sigma)}, \tag{4.1d}$$

$$\lambda^{\alpha(2)} = \sum_{n=-\infty}^{\infty} \tilde{d}^\alpha_n e^{-2in(\tau+\sigma)}, \tag{4.1e}$$

$$\chi^{a(1)} = \sum_{n=-\infty}^{\infty} S^a_n e^{-2in(\tau-\sigma)}, \tag{4.1f}$$

$$\chi^{a(2)} = \sum_{n=-\infty}^{\infty} \tilde{S}^a_n e^{-2in(\tau+\sigma)}, \tag{4.1g}$$

* The light-cone coordinates on the string world sheet are defined by $\xi^\pm = \sqrt{\frac{1}{2}}(\tau \pm \sigma)$, and those on M_d by $x^\pm = \sqrt{\frac{1}{2}}(x^0 \pm x^{d-1})$. In this section the index i refers to transverse directions on M_d

where $\lambda^{\alpha(1)}(\lambda^{\alpha(2)})$ is the single non-vanishing component of $\lambda^{\alpha}_{L}(\lambda^{\alpha}_{R})$. In (4.1d) and (4.1e) the sums are over integers (half integers) if a periodic (antiperiodic) boundary condition for λ is chosen, and similarly for (4.1f) and (4.1g). To eliminate the gauge degrees of freedom, we use the light-cone gauge defined by

$$x^{+} = P^{+}\tau\,, \qquad \lambda^{+(1)} = 0\,, \qquad \lambda^{+(2)} = 0\,. \tag{4.2}$$

In this gauge, substituting (4.1) into the constraint equations (3.14) and $++$ and $--$ projections of (3.15), we solve for the Fourier components of x^{-} and λ^{-}, respectively, as follows:

$$\alpha_n^- = \frac{2}{p^+}\left(\tfrac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^i_{n-m}\alpha^i_m + \tfrac{1}{2}\sum_{m=-\infty}^{\infty} \beta^a_{n-m}\beta^a_m \right.$$
$$\left. + \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\left(m-\tfrac{1}{2}n\right) d^i_{n-m} d^i_m + \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\left(m-\tfrac{1}{2}n\right) S^a_{n-m} S^a_m \right) \equiv \frac{2}{p^+} L_n\,, \tag{4.3a}$$

$$\tilde{\alpha}_n^- = \frac{2}{p^+}\tilde{L}_n\,, \qquad p^- = 2\alpha_0^- = 2\tilde{\alpha}_0^-\,, \qquad p^i = 2\alpha_0^i = 2\tilde{\alpha}_0^i\,,$$

$$d_n^- = \frac{2}{p^+}\left(\sum_{m=-\infty}^{\infty} d^i_{n-m}\alpha^i_m + \sum_{m=-\infty}^{\infty} S^a_{n-m}\beta^a_m - \frac{1}{6\sqrt{k}}\, i f_{abc} \sum_{l,m=-\infty}^{\infty} S^a_{n-m-l} S^b_l S^c_m \right)$$
$$\equiv \frac{2}{p^+}F_n\,, \tag{4.3b}$$

$$\tilde{d}_n^- = \frac{2}{p^+}\tilde{F}_n\,,$$

where $\tilde{L}_n$ and $\tilde{F}_n$ are the same as L_n and F_n with all the oscillators replaced by the ones with tilde. We now quantize this system and compute the central extension [7, 8a] in the commutator algebra of L_n and F_n. Demanding the closure of the Lorentz algebra requires a special value of this central extension which in turn determines the critical dimension. The quantization proceeds by imposing the following (anti) commutator relations

$$\left[q^i, p^j\right] = i\,\delta^{ij}\,, \qquad \left[\alpha^i_m, \alpha^j_n\right] = m\,\delta_{m+n,o}\,\delta^{ij}\,, \tag{4.4a}$$

$$\left[\beta^a_m, \beta^b_n\right] = \frac{-i}{\sqrt{k}} f^{abc}\beta^c_{m+n} + n\,\delta_{m+n,o}\,\delta^{ab}\,, \tag{4.4b}$$

$$\left\{d^i_m, d^j_n\right\} = \delta^{ij}\,\delta_{m+n,o}\,, \tag{4.4c}$$

$$\left\{S^a_m, S^b_n\right\} = \delta^{ab}\,\delta_{m+n,o}\,. \tag{4.4d}$$

Similar relations hold for tilde oscillators, and the latter commute with those without tilde. In deriving (4.4b) we have used the result of Witten [9] which gives the commutator $[J^a_{+L}, J^b_{+L}]$.

We now consider the commutator algebra of L_n and F_n given in (4.3). First we note that these satisfy the following (anti) Poisson brackets:

$$\begin{aligned}
[L_m, L_n]_P &= (m-n) L_{m+n} ,\\
[F_m, L_n]_P &= \left(m - \tfrac{1}{2}n\right) F_{m+n} ,\\
\{F_m, F_n\}_P &= 2L_{m+n}
\end{aligned} \tag{4.5}$$

and similar expressions for $\tilde{L}_m$ and $\tilde{F}_m$.

These brackets define a super Virasoro algebra with no central extension. In the quantum case one must take care of operator ordering. With the normalization of the individual terms given as in (4.3) one finds, however, that the (anti) commutator algebra of L_n and F_n does not close. By demanding closure, we find that the terms in L_m and F_m have to be normalized differently than in (4.3). The resulting (quantum) expressions are

$$\begin{aligned}
L_n = {} & \tfrac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha^i_{n-m} \alpha^i_m : \\
& + \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \left(m - \tfrac{1}{2}n\right) : d^i_{n-m} d^i_m : + \frac{1}{2(1 + c_A/2k)} \sum_{m=-\infty}^{\infty} : \beta^a_{n-m} \beta^a_m : \\
& + \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \left(m - \tfrac{1}{2}n\right) : S^a_{n-m} S^a_m : + \left(\varepsilon(d-2) + \varepsilon' d_G\right) \delta_{n,o} ,
\end{aligned} \tag{4.6a}$$

$$\begin{aligned}
F_n = {} & \sum_{m=-\infty}^{\infty} : d^i_{n-m} \alpha^i_m : + \frac{1}{\sqrt{1 + c_A/2k}} \\
& \times \left[\sum_{m=-\infty}^{\infty} : \beta^a_{n-m} S^a_m : - \frac{i}{6\sqrt{k}} f^{abc} \sum_{l,m=-\infty}^{\infty} : S^a_{n-l-m} S^b_m S^c_l : \right] ,
\end{aligned} \tag{4.6b}$$

where

$$\begin{aligned}
& p^- = \frac{4}{p+} (L_0 - \alpha_0) = \frac{4}{p+} (\tilde{L}_0 - \alpha_0) ,\\
& f_{acd} f_b{}^{cd} = c_A \delta_{ab} ,\\
& \varepsilon = \begin{cases} 0 , & \text{antiperiodic b.c. for } \lambda \\ \frac{1}{16} , & \text{periodic b.c. for } \lambda , \end{cases} \\
& \varepsilon' = \begin{cases} 0 , & \text{antiperiodic b.c. for } \chi \\ \frac{1}{16} , & \text{periodic b.c. for } \chi . \end{cases}
\end{aligned} \tag{4.7}$$

In (4.6a) in order to discuss all the boundary conditions simultaneously we have added a constant term to L_0. For the value of c_A, see eq. (1.3). One can now show that L_n and F_n satisfy the super-Virasoro algebra [16]

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}cm(m^2-1)\delta_{m+n,o}, \tag{4.8a}$$

$$\{F_m, F_n\} = 2L_{m+n} + \tfrac{1}{3}c\left(m^2 - \tfrac{1}{4}\right)\delta_{m+n,o}, \tag{4.8b}$$

$$[F_m, L_n] = \left(m - \tfrac{1}{2}n\right)F_{m+n}, \tag{4.8c}$$

where c is the central extension which is given by

$$c = \tfrac{3}{2}(d-2) + \frac{d_G}{1 + c_A/2k} + \tfrac{1}{2}d_G. \tag{4.9}$$

We now require Lorentz invariance in M_d. The only non-trivial commutator is $[M^{i-}, M^{j-}] = 0$. We find that this commutator holds provided that $\alpha_0 = \frac{1}{2}$ and $c = 12$. Thus follows the following critical dimension formula:

$$d = 10 - \frac{2}{3}\frac{d_G}{1 + c_A/2k} - \tfrac{1}{3}d_G \qquad \text{(spinning string)}. \tag{4.10}$$

In the case of the purely bosonic string where d_m^i and S_m^a are absent (and thus $F_m = 0$), the only surviving commutator is given by (4.8a) with central extension c which now reads

$$c = d - 2 + \frac{d_G}{1 + c_A/2k} \qquad \text{(bosonic string)}. \tag{4.11}$$

The validity of $[M^{i-}, M^{j-}] = 0$ now requires that $\alpha_0 = 1$ and $c = 24$ which implies the following critical dimension formula:

$$d = 26 - \frac{d_G}{1 + c_A/2k} \qquad \text{(bosonic string)}. \tag{4.12}$$

Finally, we quote the quantum mass formula

$$M^2 = 8(L_0 - \alpha_0) - p^i p^i. \tag{4.13}$$

For the case of semisimple group $G = G_1 \times G_2 \times \cdots \times G_P$ the formulae (4.10) and (4.12) are replaced by

$$d = 10 - \tfrac{2}{3}\sum_{i=1}^{P}\frac{d_G^{(i)}}{1 + c_A^{(i)}/2k^{(i)}} - \tfrac{1}{3}\sum_{i=1}^{P} d_G^{(i)} \qquad \text{(spinning string)} \tag{4.14}$$

$$d = 26 - \sum_{i=1}^{P}\frac{d_G^{(i)}}{1 + c_A^{(i)}/2k^{(i)}} \qquad \text{(bosonic string)}. \tag{4.15}$$

5. Mass spectrum

In this section we study the mass spectrum of a bosonic *closed* string moving on $(\text{Minkowski})_d \times \text{G}$, which will illustrate both gravitational as well as Yang-Mills degrees of freedom. We shall comment on the spinning string case at the end of this section.

The quantum mass operator in $(\text{Minkowski})_d$ of the closed bosonic string is given by (see eq. (4.13))

$$\tfrac{1}{4}\alpha' M^2 = N_d + N_{\text{G}} + 1\,, \tag{5.1}$$

where

$$N_d = \sum_{i=1}^{d-2} \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n\,, \tag{5.2}$$

$$N_{\text{G}} = \frac{1}{2(1+c_{\text{A}}/2k)} \sum_{a=1}^{d_{\text{G}}} \sum_{n=-\infty}^{\infty} :\beta^a_{-n}\beta^a_n:\,. \tag{5.3}$$

N_d and N_{G} are the occupation number operators for the left movers satisfying $[N_d, \alpha^i_n] = -n\alpha^i_n, [N_{\text{G}}, \beta^a_n] = -n\beta^a_n$. Analogously one defines the occupation number operators for the right movers, $\tilde{N}_d$ and $\tilde{N}_{\text{G}}$ with α and β replaced by $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. These number operators must satisfy the following closed string constraint:

$$N_d + N_{\text{G}} = \tilde{N}_d + \tilde{N}_{\text{G}}\,. \tag{5.4}$$

To identify the mass spectrum we need to study the representations of the Kac-Moody algebra [20] of the operators β^a_n. (For a review see ref. [21].) From now on, we shall specialize to the case of simply laced G. (The more complicated case of non-simply laced algebras has been treated in ref. [21].) Furthermore we restrict ourselves to $k=1$. As for the ordinary Lie algebras every representation is characterized by a highest weight vector. The basis of the representation space is obtained by successive application of the step operators on it. A weight vector obtained in this way is denoted by [20]

$$l = (\Lambda^I, \kappa, \delta)\,, \quad I = 1,\ldots,\text{rank}\,\text{G}\,, \tag{5.5}$$

where Λ^I are the components of a vector on the weight lattice of G, κ is the eigenvalue of k (see (4.4b)) and δ is the eigenvalue of the derivation operator d [20] which, in our case, is nothing but $(1 - N_{\text{G}})$.

We shall consider the basic representation [20] which is characterized by the highest weight $l_0 = (0,1,1)$. This weight corresponds to the ground state $|0\rangle$. Given the highest weight l_0, the remaining bases of the representation space are obtained by successive application of the appropriate step operators, $\beta^a_{-n} \ldots \beta^b_{-m}|0\rangle$ with positive $n, \ldots, m$. Using the infinite discrete Weyl group, $\hat{W}$, Frenkel and Kac [29] have shown that the weight vectors of the states obtained in this way must have the form (proposition (2.1) in ref. [29])

$$\left(\alpha^I, 1, 1 - p - \tfrac{1}{2}\alpha^I\alpha^I\right), \qquad p = 0, 1, 2, \ldots, \tag{5.6}$$

with degeneracy $M_p(r)$ given by

$$\sum_{p=0}^{\infty} M_p(r) x^p = \prod_{p=1}^{\infty} (1 - x^p)^{-r} = 1 + rx + \tfrac{1}{2}r(r+3)x^2 + \cdots . \tag{5.7}$$

Here r is the rank of G, and α^I is a vector on the root lattice. From (5.6) it follows that $N_G = p + (\frac{1}{2}\alpha^I\alpha^I)$. Thus, eq. (5.1) implies

$$\tfrac{1}{4}\alpha' M^2 = p + \tfrac{1}{2}\alpha^I\alpha^I - 1 . \tag{5.8}$$

To illustrate the details of the spectrum let us consider the case of $M_{24} \times SU(3)$ as an example. The states $\beta^a_{-n_1}\beta^b_{-n_2} \ldots \beta^c_{-n_l}|0\rangle$ are eigenstates of $d = 1 - N_G$ with eigenvalues $1 - (n_1 + n_2 + \cdots + n_l)$. Hence the states $\beta^a_{-1}|0\rangle$ have $d = 0$. These have the weights $(\alpha, 1, 0)$ and $(0, 1, 0)$ where α is a root of SU(3). The six weights $(\alpha, 1, 0)$ have unit multiplicity each, while the multiplicity of $(0, 1, 0)$ is two. Therefore they form an SU(3) octet.

The next set of states (corresponding to $d = -1$) are $\beta^a_{-1}\beta^b_{-1}|\rangle$ and $\beta^a_{-2}|0\rangle$. By virtue of the commutation relations satisfied by the β's, the state $\beta^a_{-2}|0\rangle$ is the antisymmetric part of $\beta^a_{-1}\beta^b_{-1}|0\rangle$. Thus this state is not independent.

The states $\beta^a_{-1}\beta^b_{-1}|0\rangle$ have the weights $(\alpha, 1, -1)$ and $(0, 1, -1)$ with multiplicities 2 and 5, respectively. They therefore form the representation $1 \oplus 8 \oplus 8$ of SU(3). Note that the states 10, $\overline{10}$ and 27 in the product 8×8 are not present because their weights are not of the form given in (5.6). One can show that they have vanishing norm. We have checked that the spectrum thus obtained indeed does coincide with the mass spectrum of a bosonic string moving on a direct product of 24-dimensional Minkowski space with a 2-dimensional torus. As mentioned in the introduction, various aspects of this equivalence have been discussed in refs. [7, 8a, 19, 20].

To apply this construction to the case of a closed bosonic string propagating on $M_d \times G$ we need to include $\tilde{\beta}^a_n$ as well as α^i_n and $\tilde{\alpha}^i_n$ in the operator algebra. Since β and $\tilde{\beta}$ generate two commuting Kac-Moody algebras the spectrum has $G \times G$ symmetry.

The ground state $|0\rangle$ here is a singlet of the Poincaré group of M_d as well as of $G \times G$. It is annihilated by all α^i_n, $\tilde{\alpha}^i_n$, β^a_n and $\tilde{\beta}^a_n$ for which $n \geqslant 1$. Hence its mass is given by $\frac{1}{4}\alpha' M^2 = -1$ and therefore it is a tachyonic state.

The first excited level is massless and consists of the following states: $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$ containing a graviton, a second-rank antisymmetric tensor and a dilaton; the states

TABLE 1
The scalars obtained by the operation of β^{a}_{-n} and $\tilde{\beta}^{a}_{n}$ on $|0\rangle$ for the first five levels for G = SU(3)

$\frac{1}{4}\alpha' M^2$	States	SU(3) × SU(3) representation
-1	$\|0\rangle$	(1,1)
0	$\beta^{a}_{-1}\tilde{\beta}^{b}_{-1}\|0\rangle$	(8,8)
$+1$	$\beta^{a}_{-1}\beta^{b}_{-1}\tilde{\beta}^{c}_{-1}\tilde{\beta}^{d}_{-1}\|0\rangle$	(1,1) + 2(1,8) + 2(8,1) + 4(8,8)
$+2$	$\beta^{a_1}_{-1}\beta^{\cdot}_{-1}\,\beta^{\cdot}_{-1}\,\tilde{\beta}^{\cdot}_{-1}\,\tilde{\beta}^{\cdot}_{-1}\,\tilde{\beta}^{a_6}_{-1}\|0\rangle$	4(1,1) + 6(1,8) + 6(8,1) + 2(1,10) + 2(10,1)
		+ 2(1, $\overline{10}$) + 2($\overline{10}$,1) + 9(8,8) + 3(8,10) + 3(10,8)
		+ 3(8, $\overline{10}$) + 3($\overline{10}$,8) + (10,10) + (10, $\overline{10}$)
		+ ($\overline{10}$,10) + ($\overline{10}$, $\overline{10}$)
$+3$	$\beta^{a_1}_{-1}\beta^{\cdot}_{-1}\,\beta^{\cdot}_{-1}\,\beta^{\cdot}_{-1}$	(1,1) + 6(1,8) + 6(8,1) + (1,10) + (10,1) + (1, $\overline{10}$)
	$\times\tilde{\beta}^{\cdot}_{-1}\,\tilde{\beta}^{\cdot}_{-1}\,\tilde{\beta}^{\cdot}_{-1}\,\tilde{\beta}^{a_8}_{-1}\|0\rangle$	+ ($\overline{10}$,1) + (1,27) + (27,1) + 36(8,8)
		+ 6(8,10) + 6(10,8) + 6(8, $\overline{10}$) + 6($\overline{10}$,8) + 6(8,27)
		+ 6(27,8) + (10,10) + (10, $\overline{10}$) + ($\overline{10}$,10) + ($\overline{10}$, $\overline{10}$)
		+ (10,27) + (27,10) + ($\overline{10}$,27) + (27, $\overline{10}$) + (27,27)

$\alpha^{i}_{-1}\tilde{\beta}^{a}_{-1}|0\rangle$ and $\tilde{\alpha}^{i}_{-1}\beta^{a}_{-1}|0\rangle$ which have spin one and transform in the adjoint representation of G × G; and finally there are the scalars $\beta^{a}_{-1}\tilde{\beta}^{a}_{-1}|0\rangle$ which transform as (adj, adj) of G × G. For G = SU(3) in table 1 we have given the spin-zero states obtained by the operation of β and $\tilde{\beta}$ on $|0\rangle$ for the first five levels.

For the closed spinning string the mass formula reads:

$$\tfrac{1}{4}\alpha' M^2 = N^{d}_{\mathrm{B}} + N^{\mathrm{G}}_{\mathrm{B}} + N^{d}_{\mathrm{F}} + N^{\mathrm{G}}_{\mathrm{F}} + \Delta\,,$$

where

$$N^{d}_{\mathrm{F}} = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} n : d^{i}_{-n} d^{i}_{n} :\,,$$

$$N^{\mathrm{G}}_{\mathrm{F}} = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} n : S^{a}_{-n} S^{a}_{n} :$$

and N^{d}_{B} and $N^{\mathrm{G}}_{\mathrm{B}}$ are given as before with the constraint

$$N^{d}_{\mathrm{B}} + N^{\mathrm{G}}_{\mathrm{B}} + N^{d}_{\mathrm{F}} + N^{\mathrm{G}}_{\mathrm{F}} = \tilde{N}^{d}_{\mathrm{B}} + \tilde{N}^{\mathrm{G}}_{\mathrm{B}} + \tilde{N}^{d}_{\mathrm{F}} + \tilde{N}^{\mathrm{G}}_{\mathrm{F}}\,.$$

Corresponding to periodic (P) and antiperiodic (AP) boundary conditions, there are nine possible sectors. However to simplify the discussion we consider only four of these sectors corresponding to the cases where $\lambda^{(1)}$ and $\lambda^{(2)}$ (idem $\chi^{(1)}$ and $\chi^{(2)}$) obey the same boundary conditions. This will not change our results. On table 2 we give Δ as well as the transformation properties of the ground state and the masses of

TABLE 2
The values of Δ, the transformation properties of the ground state and the mass levels for different sectors, corresponding to periodic (P) or antiperiodic (AP) boundary conditions or the fermions λ^α and χ^a

λ^α	χ^a	Δ	ground state	$\frac{1}{4}\alpha' M^2$
AP	AP	$-\frac{1}{2}$	scalar of M_d and G	$-\frac{1}{2}, 0, \frac{1}{2}, .$
AP	P	$\frac{1}{16}(d_G - 8)$	scalar of M_d and spinor of $SO(d_G)$	$\frac{1}{16}(d_G - 8), \frac{1}{16}d_G, \frac{1}{16}(d_G + 8), \ldots$
P	AP	$\frac{1}{16}(d - 10)$	spinor of M_d scalar of G^d	$\frac{1}{16}(d-10), \frac{1}{16}(d-2), \frac{1}{16}(d+6),\ .$
P	P	$\frac{1}{16}(d + d_G - 10)$	spinor of M_d and $SO(d_G)$	$\frac{1}{16}(d + d_G - 10), \frac{1}{16}(d + d_G + 6), \frac{1}{16}(d + d_G + 22), ..$

the first few levels. We can infer from this table that if we desire to have a non-abelian group G we cannot have massless fermions, in agreement with the result of Friedan and Shenker [22]. To clarify this, note that physical fermions must have periodic boundary conditions in Minkowski space-time and therefore are contained in the last two rows of table 2. Thus, supersymmetry in space-time (as contrasted to supersymmetry in $d = 2$) is unlikely to arise in such theories. The only possibility for massless fermions is when $d + d_G = 10$, which when combined with eq. (1.2) gives the unique solution $c_A = 0$. This would permit only a product of U(1)'s (e.g. $U(1)^4$ for $d = 6$)*.

6. Open problems

The results of this paper can be extended in the following directions:

(i) Generalization of our model to spaces other than group manifolds.

(ii) Generalization to $N = 2, 4, 8, 16$ supersymmetries in $d = 2$. The extension of our $N = 1$ model to $N = 2$ has been recently obtained [23], though the critical dimensional formula has not been derived. (See however ref. [24].)

(iii) To study, in any one of the models mentioned above, the possibility of introducing consistent interactions.

In this paper, in d-dimensional space-time we have lost supersymmetry. To maintain it, one may have to consider models of the type constructed by Green and Schwarz [18]. Such models can exist in 3, 4, 6 and 10 dimensions. Recently, Witten has constructed the $d = 10$ Green-Schwarz type action in curved space [25]. Follow-

* Note that there are two possible G-parity [15] operators in the theories of this type, $(-1)^{2N_F^d - 1}$ or $(-1)^{2N_F^{G'} - 1}$ These may be useful for suppressing the tachyons

ing Witten's method, we have constructed similar actions in $d = 3, 4, 6$ [26]. In this context it is worth remarking that an anomaly-free $N = 2$, $d = 6$ supergravity with $E_6 \times E_7 \times U(1)$ symmetry exists [17] and may provide a good phenomenological model for quarks and leptons. Whether such a theory is a field theoretical limit of a string theory remains to be seen.

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References

[1] S. Deser and B Zumino, Phys. Lett. 65B (1976) 369
[2] L. Brink, P. di Vecchia and P Howe, Phys Lett 65B (1976) 471
[3] A. Neveu and J H Schwarz, Nucl. Phys. B31 (1971) 86; Phys Rev. D4 (1971) 1109,
P Ramond, Phys Rev D3 (1971) 2415
[4] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95;
E. Witten, Nucl. Phys. B223 (1983) 422, 433
[5] S.J Gates, Jr, C M. Hull and M Roček, Nucl Phys B248 (1984) 157
[6] T Curtright and C Zachos, Phys Rev Lett 53 (1984) 1799,
R. Rohm, Anomalous interactions for the supersymmetric nonlinear σ-model in two dimensions, Princeton preprint (October 1984);
E Braaten, T.L Curtright and C K Zachos, Nucl. Phys. B260 (1985) 619
[7] D. Nemeschansky and S Yankielowicz, Critical dimension of string theories in curved space, Phys Rev. Lett. 54 (1985) 260
[8a] P.G Freund, Santa Barbara preprint (1984);
V G Knizhnik and A.B Zamolodchikov, Nucl. Phys. B247 (1984) 83,
V G. Kac, Infinite dimensional Lie algebras, *in* Prog. in Maths. vol 44 (Birkhäuser, Boston, 1983) p 245;
P Goddard, D. Olive and A. Schwimmer, Imperial College preprint TP/84-85/23;
P Goddard, W. Nahm and D Olive, Symmetric spaces and chiral field theories in two dimensions, Imperial College preprint (1985)
[8b] S Jain, R Shankar and S.R Wadia, Conformal invariance and string theories in compact space: bosons, Phys Rev. D32 (1985) 2713
[9] E Witten, Comm Math Phys 92 (1984) 455
[10] I Antoniadis and C Bachas, Conformal invariance and parastatistics in two dimensions, SLAC preprint SLAC-PUB-3625 (April 1985)
[11] F. Ardalan and F. Mansouri, Phys Rev D9 (1974) 3341;
J.F L. Hopkinson and R W. Tucker, Phys Rev D10 (1974) 558
[12] D.J. Gross, J.A. Harvey, E. Martinec and R Rohm, Nucl Phys B256 (1985) 253; B267 (1986) 75
[13] L Alvarez-Gaumé and D.Z. Freedman, Comm Math. Phys 80 (1981) 443
[14] P. Forgács and N S. Manton, Comm Math. Phys. 72 (1980) 15
[15] J H. Schwarz, Phys Reports 89, (1982) 223
[16] P. Goddard and D. Olive, Nucl Phys B257 [FS14] (1985) 226
[17] S Randjbar-Daemi, A. Salam, E Sezgin and J. Strathdee, Phys Lett. 151B (1985) 351,
A Salam and E. Sezgin, ICTP, Trieste, preprint IC/85/19 (1985)
[18] M.B. Green and J.H. Schwarz, Nucl Phys B181 (1981) 502
[19] I. Frenkel and V G. Kac, Inventiones Math. 62 (1980) 23
[20] D Olive, Kac-Moody algebras: An introduction for physicists, Imperial College preprint, London (March 1985);

P Goddard, Infinite-dimensional Lie algebras· representations and applications, Cambridge preprint (1985)
[21] A J. Feingold and I Frenkel, Classical affine algebras, preprint
[22] D. Friedan and S. Shenker, Chicago preprint (1985)
[23] E. Bergshoeff, H Nishino and E. Sezgin, Phys Lett 166B (1986) 242
[24] C M Hull, σ-model beta-functions and string compactifications, Cambridge preprint (1985)
[25] E Witten, Twistor-like transform in ten dimensions, Princeton preprint (May 1985)
[26] E Bergshoeff, E Sezgin and P.K. Townsend, Superstring actions in $d = 3, 4, 6, 10$ curved superspace, Trieste preprint (1985), to appear in Phys Lett B